Resonant Frequency of Open-Ended Cylindrical Cavity

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Abstract—**The TE_{011} mode of oscillation in an open-ended circular cylindrical microwave cavity is analyzed. The cavity consists of a circular waveguide that is terminated at each end with a thin cylindrical partition coaxial with the circular waveguide. The resonant frequency of the cavity is computed by using Laplace transform and Wiener-Hopf techniques. Numerical values for the resonant frequency are presented.**

Introduction

MICROWAVE cavities have been used extensively to measure the dielectric properties of liquids and gases. The change in resonant frequency and cavity Q when a material is introduced into a microwave cavity gives a measure of the complex dielectric constant of the material [1]. In many applications, such as in atmospheric research, it is necessary that the cavity be open-ended so that dynamic measurements of the dielectric properties can be made. This requires replacing the solid end walls of the cavity by a termination that will totally reflect the microwave energy and yet present a minimal obstruction to the flow of material through the cavity. A discussion about various types of terminations may be found in references [2]–[5].

Figure 1 shows a type of open-ended cavity that is frequently used. It consists of a circular waveguide that is terminated at each end with a coaxial cylindrical partition. This partition separates a portion of the waveguide into a coaxial waveguide plus a smaller circular waveguide that also serves as the inner conductor for the coaxial waveguide. The partition will act like a perfect reflector to the microwave energy if the cavity dimensions and frequency are selected so that the modes excited in the coaxial waveguide and smaller circular waveguide are cutoff modes.

This paper presents an analysis of the TE_{011} mode of oscillation in open-ended cavities of the type shown in Fig. 1 to determine the relation between the resonant frequency of the cavity and the cavity dimensions. The analysis is restricted to the TE_{011} mode since this mode of oscillation has a very high Q and therefore is commonly used. The solution for the resonant frequency of this mode can be easily computed if the reflection coefficient of the TE_{01} circular waveguide mode incident on the cylindrical partition is known.

The model that will be used to compute the reflection coefficient is shown in Fig. 2. The various regions of interest have been numbered for ease in reference. The model consists of a perfectly conducting circular waveguide of radius b and of infinite extent in the z-direction. Coaxial with the waveguide is an infinitely thin, perfectly conducting circular waveguide of radius a that extends over the range $0<z<\infty$. The reflection coefficient will be computed for the case of the TE_{01} mode incident from the left.

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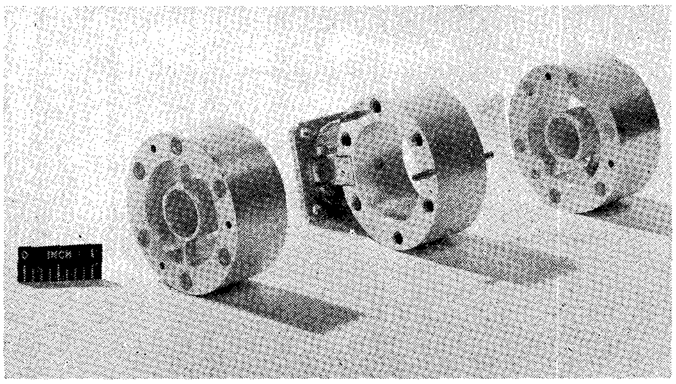

Fig. 1. Open-ended microwave cavity.

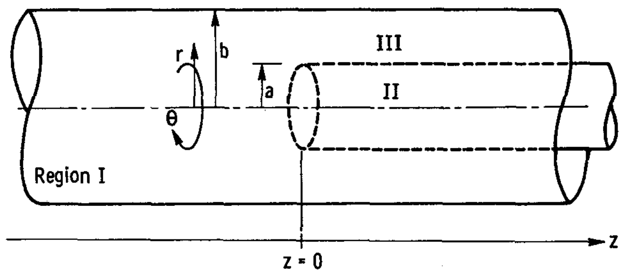

Fig. 2. Cylindrical partition in waveguide.

Before proceeding with the analysis, it will be instructive to consider the various types of waves that can exist in the different regions. It will only be necessary, however, to consider the circularly symmetric TE_{0n} modes in each region since both the TE_{01} incident mode and the model possess circular symmetry. Thus, the field components of interest will be the θ-component of the electric field and the r- and z-components of the magnetic field. Since the electric and magnetic fields are related by Maxwell's equations, only the electric field needs to be determined to specify the total field uniquely.

Waveguide Modes

Solutions for the electric field $E_\theta(r, z)$ in the various regions of interest are [6]

$$E_\theta(r, z) = J_1(\Gamma_1 r)e^{-j\beta_1 z} + RJ_1(\Gamma_1 r)e^{j\beta_1 z} + \sum_{n=2}^{\infty} A_n J_1(\Gamma_n r)e^{\beta_n z} \quad \text{Region I} \qquad (1a)$$

$$E_\theta(r, z) = \sum_{n=1}^{\infty} B_n J_1(\gamma_n r) e^{-\alpha_n z} \quad \text{Region II} \qquad (1b)$$

$$E_\theta(r, z) = \sum_{n=1}^{\infty} C_n [N_1(\delta_n a) J_1(\delta_n r) - J_1(\delta_n a) N_1(\delta_n r)] e^{-\rho_n z} \quad \text{Region III} \qquad (1c)$$

where the A_n, B_n, and C_n are complex amplitude constants, and R is the reflection coefficient for the TE_{01} mode. A time dependence of $e^{j\omega t}$ is implicit in these equations. Since the solutions must satisfy Maxwell's equations, the propagation constants β_n, α_n, and ρ_n must satisfy the equations

$$\left.\begin{aligned} \beta_1^2 &= k_0^2 - \Gamma_1^2 \\ \beta_n^2 &= \Gamma_n^2 - k_0^2, \qquad n > 1 \end{aligned}\right\} \qquad (2a)$$

$$\alpha_n^2 = \gamma_n^2 - k_0^2 \qquad (2b)$$

$$\rho_n^2 = \delta_n^2 - k_0^2 \qquad (2c)$$

where $k_0 \equiv \omega/c$.

It will be assumed that k_0 satisfies the inequalities $\Gamma_1 < k_0 < \Gamma_2$, $k_0 < \gamma_1$, and $k_0 < \delta_1$ so that the TE_{01} mode will propagate in Region I, and that the TE_{0n}, $n > 1$ modes in Region I and all TE_{0n} modes in Regions II and III will be cutoff modes.

The eigenvalues Γ_n, γ_n, and δ_n are determined by the boundary condition that requires E_θ to vanish on the perfectly conducting surfaces $r = b$ and $r = a$ for $z > 0$ (see Fig. 2). Thus,

$$J_1(\Gamma_n b) = 0, \ \Gamma_1 b = 3.8317, \ \Gamma_2 b = 7.0156, \cdots, \Gamma_n b \approx (n + 1/4)\pi$$

$$J_1(\gamma_n a) = 0, \ \gamma_1 a = 3.8317, \ \gamma_2 a = 7.0156, \cdots, \gamma_n a \approx (n + 1/4)\pi$$

$$N_1(\delta_n a) J_1(\delta_n b) - J_1(\delta_n a) N_1(\delta_n b) = 0, \cdots, \delta_n a \approx \frac{n\pi}{b/a - 1}.$$

Solution for Reflection Coefficient

The formal solution for the reflection coefficient R will be obtained by setting up an integral equation for the scattered electric field due to the TE_{01} mode incident on the cylindrical partition and then solving this equation by using the Wiener-Hopf technique. The reflection coefficient can also be determined by equating the general solutions for the electromagnetic field at the surface $z = 0$ (see Fig. 2) and then solving the resulting equations for the amplitudes A_n, B_n, C_n, and R by using a function-theoretic technique [7], [8].

The integral equation will be formulated by employing the Green's function, $G(r, a, z - z_0)$, which is defined to be the solution of the differential equation

$$\frac{\partial^2 G}{\partial r^2} + \frac{1}{r}\frac{\partial G}{\partial r} + \frac{\partial^2 G}{\partial z^2} + \left(k_0^2 - \frac{1}{r^2}\right) G = j\omega\mu_0 \delta(z - z_0)\delta(r - a) \qquad (3a)$$

which satisfies the boundary condition

$$G(b, a, z - z_0) = 0. \qquad (3b)$$

Physically, the Green's function corresponds to the θ-component of the electric field in an infinitely long circular waveguide of radius b which is produced by a filamentary current loop of radius a located at $z = z_0$. Thus it must follow that the total electric field in the waveguide E_θ is given by

$$E_\theta(r, z) = J_1(\Gamma_1 r) e^{-j\beta_1 z} + \int_0^\infty G(r, z, z - z_0) J_\theta(a, z_0) dz_0 \qquad (4)$$

where $J_\theta(a, z_0)$ is the electric current density on the cylindrical partition. The first term on the right in (4) corresponds to the incident TE_{01} mode, which will be defined to exist for all values of z, and the second term corresponds to the field produced by the induced electric current on the cylindrical partition.

It will be convenient in the following analysis to split the electric field E_θ into two parts—an incident field E_θ^i and a scattered field E_θ^s—so that

$$E_\theta(r, z) = E_\theta^i(r, z) + E_\theta^s(r, z) \qquad (5)$$

where

$$E_\theta^i(r, z) = J_1(\Gamma_1 r) e^{-j\beta_1 z}. \qquad (6)$$

Combining (4), (5), and (6) gives the desired integral equation for the scattered electric field E_θ^s:

$$E_\theta^s(r, z) = \int_{-\infty}^{\infty} G(r, a, z - z_0) J_\theta(a, z_0) dz_0. \qquad (7)$$

In going from (4) to (7), the lower limit on the integral was changed from 0 to $-\infty$ since $J_\theta(a, z_0)$ is zero in the range $-\infty < z_0 < 0$.

Solution of Integral Equation

The solution of the integral equation for the scattered electric field will be obtained by using Laplace transform and Wiener-Hopf techniques [9]. Let the functions $\mathcal{E}(r, \beta)$, $\mathcal{G}(r, a, \beta)$, and $\mathcal{J}(a, \beta)$ be the bilateral Laplace transforms with respect to z of $E_\theta^s(r, z)$, $G(r, a, z)$, and $J_\theta(a, z)$, respectively:

$$\mathcal{E}(r, \beta) = \int_{-\infty}^{\infty} E_\theta^s(r, z) e^{-\beta z} dz \qquad (8a)$$

$$\mathcal{G}(r, a, \beta) = \int_{-\infty}^{\infty} G(r, a, z) e^{-\beta z} dz \qquad (8b)$$

$$\mathcal{J}(a, \beta) = \int_{-\infty}^{\infty} J_\theta(a, z) e^{-\beta z} dz. \qquad (8c)$$

In order to make all the Laplace transforms exist in a common region in the complex β-plane, the propagation constant β_1 will be made complex. This is equivalent to introducing an energy loss mechanism into the medium interior to the waveguide. Let $\beta_1 = \beta_1' - j\beta_1''$ where $\beta_1' > 0$ and $\beta_1'' > 0$. In the final solution β_1'' will be set equal to zero to recover the solution for the lossless case.

The functions $\mathcal{E}$ and $\mathcal{J}$ are, at this point, unknown since E_θ^s and J_θ are unknown. However, it can be shown that $\mathcal{E}$

is analytic in the strip $-\beta_1'' < \Re e\beta < \beta_1''$ and that $\mathcal{J}$ is analytic in the region $\Re e\beta > -\beta_1''$.

The transform of the Green's function can be easily computed by first transforming the basic equation and boundary condition for the Green's function given by (3a) and (3b) and then solving the transformed equations with the result

$$\mathcal{G}(r, a, \beta) = j\omega\mu_0 a \frac{\pi}{2} \cdot \left[\frac{J_1(\lambda b)N_1(\lambda a) - J_1(\lambda a)N_1(\lambda b)}{J_1(\lambda b)} \right] J_1(\lambda r) \quad r \leq a \quad (9a)$$

$$\mathcal{G}(r, a, \beta) = j\omega\mu_0 a \frac{\pi}{2} \cdot \left[\frac{J_1(\lambda b)N_1(\lambda r) - J_1(\lambda r)N_1(\lambda b)}{J_1(\lambda b)} \right] J_1(\lambda a) \quad r \geq a \quad (9b)$$

where $\lambda \equiv (k_0^2 + \beta^2)^{1/2}$.

Since $\mathcal{G}$ is an even function of λ, either branch of λ can be selected. It can be shown that $\mathcal{G}$ is analytic in the strip $-\beta_1'' < \Re e\beta < \beta_1''$.

The Laplace transform of the integral equation (7) for the scattered electric field is given by

$$\mathcal{E}(r, \beta) = \mathcal{G}(r, a, \beta)\mathcal{J}(a, \beta) \quad (10)$$

where the faltung theorem has been used to take the transform of the integral. It will be convenient to express the transform of the scattered electric field $\mathcal{E}$ as the sum of two single-sided transforms $\mathcal{E}^+(r, \beta)$ and $\mathcal{E}^-(r, \beta)$, where

$$\mathcal{E}^+(r, \beta) = \int_0^\infty E_\theta^s(r, z)e^{-\beta z}dz \quad (11a)$$

$$\mathcal{E}^-(r, \beta) = \int_{-\infty}^0 E_\theta^s(r, z)e^{-\beta z}dz. \quad (11b)$$

The function $\mathcal{E}^-$ is, at this point, unknown. The function $\mathcal{E}^+$, however, can be computed at the radius $r = a$ by using the fact that the scattered electric field at $r = a$, for $z > 0$, must simply be the negative of the incident electric field since the total electric field must be zero to satisfy the proper boundary condition. Using the expression for the incident field given by (6) gives

$$\mathcal{E}^+(a, \beta) = -\frac{J_1(\Gamma_1 a)}{\beta + j\beta_1}. \quad (12)$$

The transform of the incident field is defined only in the region $\Re e\beta > -\beta_1''$.

Combining the result for $\mathcal{E}^+$ with (8a), (10), and (11) yields a form for the transformed integral equation that can be solved by using the Wiener-Hopf technique:

$$\mathcal{E}^-(a, \beta) - \frac{J_1(\Gamma_1 a)}{\beta + j\beta_1} = \mathcal{G}(a, a, \beta)\mathcal{J}(a, \beta). \quad (13)$$

It should be noted that when the previous equations were combined to obtain (13), the radius r had to be set equal to a in each of the equations since $\mathcal{E}^+$ could be computed only at $r = a$.

Equation (13) can be solved for $\mathcal{E}^-$ if the transformed Green's function $\mathcal{G}$ can be decomposed into the Wiener-Hopf factors $\mathcal{G}^+$ and $\mathcal{G}^-$ so that $\mathcal{G} = \mathcal{G}^+/\mathcal{G}^-$, where $\mathcal{G}^+$ is analytic and nonzero for $\Re e\beta > -\beta_1''$ and $\mathcal{G}^-$ is analytic and nonzero for $\Re e\beta < \beta_1''$. This step allows (13) to be rewritten in the form

$$\mathcal{E}^-(a, \beta)\mathcal{G}^-(a, a, \beta) + J_1(\Gamma_1 a)\frac{\mathcal{G}^-(a, a, -j\beta_1) - \mathcal{G}^-(a, a, \beta)}{\beta + j\beta_1} = \frac{J_1(\Gamma_1 a)\mathcal{G}^-(a, a, -j\beta_1)}{\beta + j\beta_1} + \mathcal{G}^+(a, a, \beta)\mathcal{J}(a, \beta). \quad (14)$$

The left side of (14) is analytic for $\Re e\beta < \beta_1''$ and the right side is analytic for $\Re e\beta > -\beta_1''$. The equality holds only in the strip $-\beta_1'' < \Re e\beta < \beta_1''$.

A function $\mathcal{F}(\beta)$ that is analytic everywhere in the finite complex β-plane can be defined from (14), where $\mathcal{F}(\beta)$ is equal to the left side of (14) for $\Re e\beta \leq -\beta_1''$, to the right side for $\Re e\beta \geq \beta_1''$, and to either side in the strip $-\beta_1'' < \Re e\beta < \beta_1''$. It can be shown that $\mathcal{F}(\beta)$ approaches zero as $\beta \to \infty$. Consequently, $\mathcal{F}(\beta)$ must be zero everywhere since zero is the only function that is analytic everywhere and vanishes at infinity (Liouville's theorem). Since $\mathcal{F}(\beta)$ is zero, each side of (14) must also be zero. Thus,

$$\mathcal{E}^-(a, \beta) = -\frac{J_1(\Gamma_1 a)\left[\mathcal{G}^-(a, a, -j\beta_1) - \mathcal{G}^-(a, a, \beta)\right]}{\mathcal{G}^-(a, a, \beta)(\beta + j\beta_1)} \quad (15a)$$

$$\mathcal{J}(a, \beta) = -\frac{J_1(\Gamma_1 a)\mathcal{G}^-(a, a, -j\beta_1)}{\mathcal{G}^+(a, a, \beta)(\beta + j\beta_1)}. \quad (15b)$$

The conditions that have been specified for $\mathcal{G}^+$ and $\mathcal{G}^-$ do not determine this pair of functions uniquely since both $\mathcal{G}^+$ and $\mathcal{G}^-$ can be multiplied by any function $p(\beta)$ that is analytic and nonzero everywhere in the finite β-plane to yield a new pair of functions that will satisfy all the original conditions. Any such pair of functions when combined with (15a) and (15b) will yield a solution for the scattered electric field which satisfies all the conditions imposed on the field up to this point.

The remaining boundary condition that must be imposed is commonly called the edge condition [10]. The edge condition requires the total electric field E_θ when evaluated at $r = a$ to be of the order $z^{1/2}$ as $z \to 0$, and the current density J_θ to be of the order $z^{-1/2}$ as $z \to 0$. These conditions require $\mathcal{G}^+$ and $\mathcal{G}^-$ to be of the order $\beta^{-1/2}$ and $\beta^{1/2}$, respectively, as $|\beta| \to \infty$.

Wiener-Hopf Factors of Green's Function

The transformed Green's function evaluated at $r = a$ is given by

$$\mathcal{G}(a, a, \beta) = j\omega\mu_0 a \frac{\pi}{2} \frac{J_1(\lambda a)}{J_1(\lambda b)} \cdot \left[J_1(\lambda b)N_1(\lambda a) - J_1(\lambda a)N_1(\lambda b)\right] \quad (16)$$

where $\lambda = (k_0^2 + \beta^2)^{1/2}$ [see (9)]. The decomposition of $\mathcal{G}$ into the ratio of $\mathcal{G}^+$ to $\mathcal{G}^-$ is best carried out by expressing $\mathcal{G}$ in the form of an infinite product [11]. Since $\mathcal{G}$ has simple zeros at $\lambda = \pm\gamma_n$ and $\lambda = \pm\delta_n$, simple poles at $\lambda = \pm\Gamma_n$, and is equal to

$$\frac{-j\omega\mu_0 a}{2}(1 - a^2/b^2) \quad \text{at} \quad \lambda = 0,$$

it can be expressed in the infinite product form

$$\mathcal{G}(a, a, \beta) = \frac{-j\omega\mu_0 a}{2}\left(1 - \frac{a^2}{b^2}\right)\frac{\prod_{n=1}^{\infty}\left(1 - \frac{\lambda^2}{\gamma_n^2}\right)\prod_{n=1}^{\infty}\left(1 - \frac{\lambda^2}{\delta_n^2}\right)}{\prod_{n=1}^{\infty}\left(1 - \frac{\lambda^2}{\Gamma_n^2}\right)}.$$

The expression for $\mathcal{G}$ can be put into a more useful form by reintroducing the parameters β_n, α_n, and ρ_n from (2a), (2b), and (2c), respectively. Thus,

$$\mathcal{G}(a, a, \beta) = \frac{+j\omega\mu_0 a}{2}\left(1 - \frac{a^2}{b^2}\right)\frac{\prod_{n=1}^{\infty}\left(\frac{\alpha_n^2 - \beta^2}{\gamma_n^2}\right)\prod_{n=1}^{\infty}\left(\frac{\rho_n^2 - \beta^2}{\delta_n^2}\right)}{\left(\frac{\beta^2 + \beta_1^2}{\Gamma_1^2}\right)\prod_{n=2}^{\infty}\left(\frac{\beta_n^2 - \beta^2}{\Gamma_n^2}\right)}. \quad (17)$$

Each of the infinite products in (17) can be easily factored into the product of two infinite products so that one infinite product is analytic and nonzero in the region $\Re e\beta < \beta_1''$ and the other is analytic and nonzero in the region $\Re e\beta > -\beta_1''$. The result of this operation allows $\mathcal{G}^+$ and $\mathcal{G}^-$ to be easily identified as

$$\mathcal{G}^+(a, a, \beta) = \frac{+j\omega\mu_0 a}{2}\left(1 - \frac{a^2}{b^2}\right) \cdot \frac{\prod_{n=1}^{\infty}\left(\frac{\alpha_n + \beta}{\gamma_n}\right)e^{-\beta a/n\pi}\prod_{n=1}^{\infty}\left(\frac{\rho_n + \beta}{\delta_n}\right)e^{-\beta(b-a)/n\pi}}{\left(\frac{\beta + j\beta_1}{\Gamma_1}\right)e^{-\beta b/\pi}\prod_{n=2}^{\infty}\left(\frac{\beta_n + \beta}{\Gamma_n}\right)e^{-\beta b/n\pi}}\, p(\beta) \quad (18a)$$

$$\mathcal{G}^-(a, a, \beta) = \frac{\left(\frac{\beta - j\beta_1}{\Gamma_1}\right)e^{\beta b/\pi}\prod_{n=2}^{\infty}\left(\frac{\beta_n - \beta}{\Gamma_n}\right)e^{\beta b/n\pi}}{\prod_{n=1}^{\infty}\left(\frac{\alpha_n - \beta}{\gamma_n}\right)e^{\beta a/n\pi}\prod_{n=1}^{\infty}\left(\frac{\rho_n - \beta}{\delta_n}\right)e^{\beta(b-a)/n\pi}}\, p(\beta). \quad (18b)$$

The quantity

$$\frac{+j\omega\mu_0 a}{2}(1 - a^2/b^2)$$

has been arbitrarily associated with the factor $\mathcal{G}^+$. The exponential terms in (18a) and (18b) were introduced to ensure that the infinite products will converge. By replacing the infinite products in (18) with their asymptotic forms for large β [7] it can be shown that a suitable choice for $p(\beta)$ which gives $\mathcal{G}^+$ and $\mathcal{G}^-$ the proper asymptotic forms is

$$p(\beta) = \exp\left[\frac{\beta a}{\pi}\ln\left(\frac{a}{b - a}\right) - \frac{\beta b}{\pi}\ln\left(\frac{b}{b - a}\right)\right].$$

Scattered Electric Field

The scattered electric field evaluated at $r = a$ can now be computed by inverting the transformed field $\mathcal{E}$:

$$E_\theta^s(a, z) = \frac{1}{2\pi j}\int_C \mathcal{E}(a, \beta)e^{\beta z}d\beta. \quad (19)$$

The inversion contour C must be located in the strip $-\beta_1'' < \Re e\beta < \beta_1''$ as shown in Fig. 3 since this is the only common region in the β-plane where all the transforms are analytic.

The transformed scattered field $\mathcal{E}(a, \beta)$ is the sum of $\mathcal{E}^+(a, \beta)$ given by (12) and $\mathcal{E}^-(a, \beta)$ given by (15a). Thus,

$$E_\theta^s(a, z) = -\frac{1}{2\pi j}\int_C \frac{J_1(\Gamma_1 a)\mathcal{G}^-(a, a, -j\beta_1)}{(\beta + j\beta_1)\mathcal{G}^-(a, a, \beta)}e^{\beta z}d\beta. \quad (20)$$

To evaluate the field for $z < 0$, the contour C can be closed in the right half β-plane with a semicircle of infinite radius. It can be easily shown that the integral over this semicircle is zero. Thus the integral over the original contour C must equal $-2\pi j$ times the sum of the residues of the poles of the integrand in the right half β-plane. The integrand has poles in this half plane at $\beta = j\beta_1$ and at $\beta = \beta_n$ ($n > 1$) owing to the zeros of $\mathcal{G}^-$. Performing the integration gives

$$E_\theta^s(a, z) = +\frac{J_1(\Gamma_1 a)\mathcal{G}^-(a, a, j\beta_1)}{2j\beta_1 \left.\frac{\partial\mathcal{G}^-}{\partial\beta}(a, a, \beta)\right|_{\beta = j\beta_1}}e^{j\beta_1 z} + \sum_{n=2}^{\infty}\frac{J_1(\Gamma_1 a)\mathcal{G}^-(a, a, -j\beta_1)}{(\beta_n + j\beta_1)\left.\frac{\partial\mathcal{G}^-}{\partial\beta}(a, a, \beta)\right|_{\beta = \beta_n}}e^{\beta_n z} \quad z < 0. \quad (21)$$

The total electric field in the region $z < 0$ can now be obtained by adding the incident field given by (6) to the scattered field given by (21) with the result

$$E_\theta(r, z) = J_1(\Gamma_1 r)e^{-j\beta_1 z} + \frac{\mathcal{G}^-(a, a, -j\beta_1)}{2j\beta_1 \left.\frac{\partial\mathcal{G}^-}{\partial\beta}(a, a, \beta)\right|_{\beta = j\beta_1}}J_1(\Gamma_1 r)e^{j\beta_1 z} + \sum_{n=2}^{\infty}\frac{\mathcal{G}^-(a, a, -j\beta_1)J_1(\Gamma_1 a)}{(\beta_n + j\beta)\left.\frac{\partial\mathcal{G}^-}{\partial\beta}(a, a, \beta)\right|_{\beta = \beta_n}}\frac{J_1(\Gamma_n r)}{J_1(\Gamma_n a)}e^{\beta_n z} \quad z < 0. \quad (22)$$

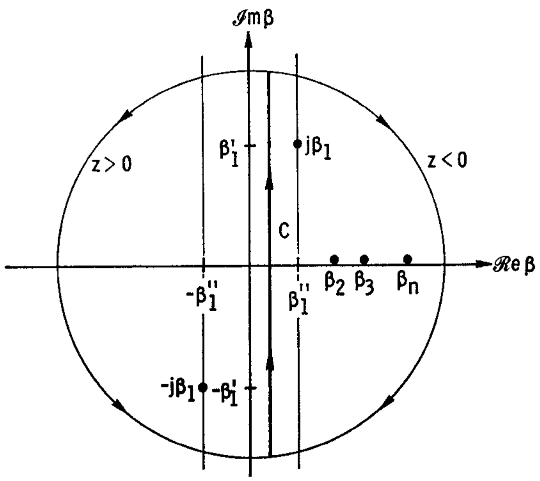

Fig. 3. Complex β-plane.

In obtaining (22), the amplitude factor of each mode in the scattered field was multiplied by $J_1(\Gamma_n r)/J_1(\Gamma_n a)$ to reintroduce the radial dependence of the modes.

If (22) is compared with the solution for the field in the region $z<0$ given by (1a), the reflection coefficient R of the TE_{01} mode can be easily identified as

$$R = \frac{\mathcal{G}^-(a, a, -j\beta_1)}{2j\beta_1 \left.\frac{\partial \mathcal{G}^-}{\partial \beta}(a, a, \beta)\right|_{\beta=j\beta_1}}. \quad (23)$$

Combining (23) with the expression for $\mathcal{G}^-$ given by (18b) allows R to be put into the form

$$R = -e^{-2j\beta_1 b/\pi} \frac{\prod_{n=2}^{\infty}\left(\frac{\beta_n + j\beta_1}{\Gamma_n}\right)e^{-j\beta_1 b/n\pi} \prod_{n=1}^{\infty}\left(\frac{\alpha_n - j\beta_1}{\gamma_n}\right)e^{j\beta_1 a/n\pi} \prod_{n=1}^{\infty}\left(\frac{\rho_n - j\beta_1}{\delta_n}\right)e^{j\beta_1(b-a)/n\pi}p(-j\beta_1)}{\prod_{n=2}^{\infty}\left(\frac{\beta_n - j\beta_1}{\Gamma_n}\right)e^{j\beta_1 b/n\pi} \prod_{n=1}^{\infty}\left(\frac{\alpha_n + j\beta_1}{\gamma_n}\right)e^{-j\beta_1 a/n\pi} \prod_{n=1}^{\infty}\left(\frac{\rho_n + j\beta_1}{\delta_n}\right)e^{-j\beta_1(b-a)/n\pi}p(j\beta_1)}. \quad (24)$$

From this point on, the propagation constant β_1 will be regarded as real so that the subsequent results will apply for the case where there are no losses.

A careful examination of the equation for R shows that apart from the first term $-\exp[-2j\beta_1 b/\pi]$, which has unit modulus, the expression for R is simply the ratio of two complex numbers, with the numerator as the complex conjugate of the denominator. Thus it must follow that $|R|=1$.

This result should be expected. The TE_{01} mode incident on the coaxial cylindrical partition shown in Fig. 2 must be perfectly reflected since it has been assumed that all the TE_{0n} modes in Regions II and III are cutoff and that there are no losses. Thus the reflected TE_{01} mode must have the same amplitude as the incident mode.

The argument or phase of R, which will be denoted by ϕ, must equal twice the phase of the numerator in (24), since the numerator is the complex conjugate of the denominator, plus π, which is a result of the initial minus sign. Thus,

$$\phi = \pi + 2\left[\sum_{n=2}^{\infty}\tan^{-1}\left(\frac{\beta_1}{\beta_n}\right) - \sum_{n=1}^{\infty}\tan^{-1}\left(\frac{\beta_1}{\alpha_n}\right) - \sum_{n=1}^{\infty}\tan^{-1}\left(\frac{\beta_1}{\rho_n}\right)\right] + 2\left[\frac{\beta_1 b}{\pi}\ln\left(\frac{b}{b-a}\right) - \frac{\beta_1 a}{\pi}\ln\left(\frac{a}{b-a}\right)\right]. \quad (25)$$

The terms in the summations are recognized as the phases of the various infinite products. The last term corresponds to twice the phase of $p(-j\beta_1)$.

Numerical values for ϕ were obtained on an IBM 7094 computer. The first 100 terms in each of the infinite series were retained. By using an integral technique it was estimated that truncating the series at 100 terms introduces an error in ϕ of less than 0.01 radian. The numerical results are shown in Figs. 4(a) and (b) in the form of curves of ϕ against $k_0 b$ for typical values of the parameter b/a. The range of $k_0 b$ is bounded. The lower limit is determined by the condition that $k_0 b$ must exceed $\Gamma_1 b$, which has the constant value 3.8317, so that the TE_{01} mode in Region I can propagate. The upper limit is determined by the condition that $k_0 b$ must be less than the smaller of $\gamma_1 b$ or $\delta_1 b$. This condition ensures that neither the TE_{01} mode in Region II nor the TE_{01} mode in Region III will propagate. The values of $\gamma_1 b$ and $\delta_1 b$ are dependent on the parameter b/a. It can be shown that, if b/a is less than 1.831, $\gamma_1 b < 7.0153 < \delta_1 b$, and if b/a exceeds 1.831, $\delta_1 b < 7.0153 < \gamma_1 b$. For the case where $b/a = 1.831$, the TE_{01} modes in Regions II and III have identical low-frequency cutoff points that correspond to the values $\gamma_1 b = \delta_1 b = 7.0153$.

The numerical results for ϕ show that at the lower limit of $k_0 b$, the value of ϕ is equal to π for all values of the parameter b/a. Thus, at the lower limit of $k_0 b$, the cylindrical partition reflects the incident TE_{01} mode as if a perfectly conducting surface spanned the waveguide cross section at $z=0$ (see Fig. 2) since a perfectly conducting surface gives a reflection coefficient of unit modulus and argument π. As $k_0 b$ increases, the incident TE_{01} mode is still perfectly reflected since the modulus of R remains at unity. However, since the phase of the reflection coefficient decreases with increasing $k_0 b$, it appears as though the position of the perfectly conducting surface moves. Thus the cylindrical partition acts like a "virtual" perfectly conducting surface that spans the entire waveguide cross section whose position is a function of $k_0 b$ and the parameter b/a.

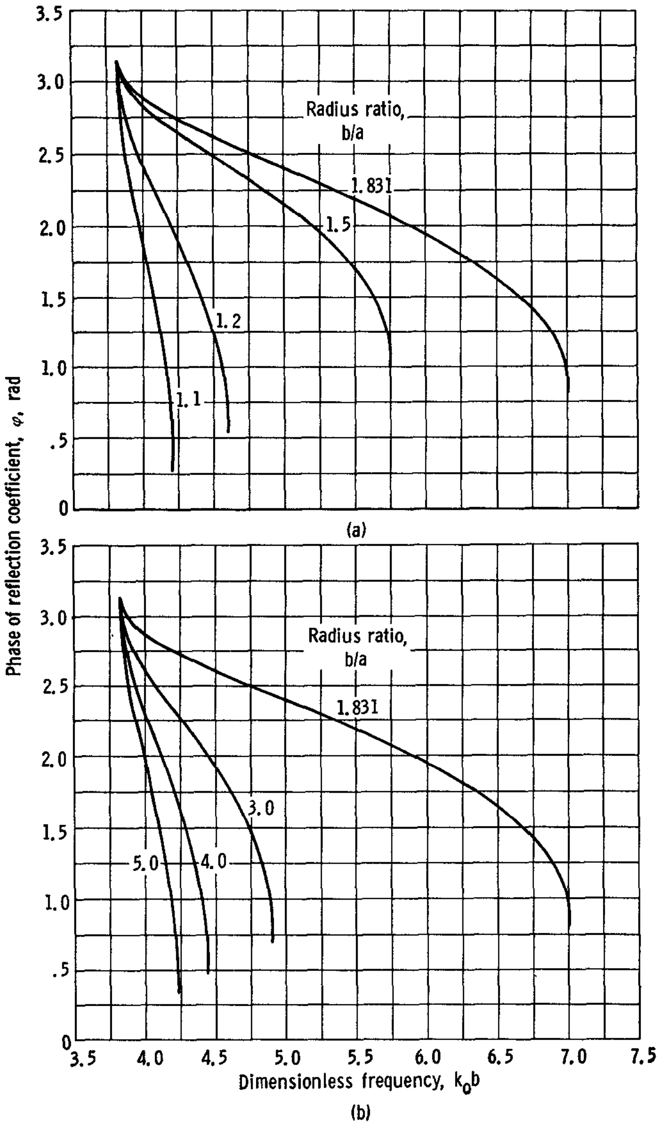

Fig. 4. Phase of reflection coefficient against dimensionless frequency for constant values of radius ratio.

Resonant Frequency of Cavity

An open-ended cavity can be constructed by placing two cylindrical partitions of finite length within a circular waveguide as shown in Fig. 5. The partitions need only be long compared with $1/\alpha_1$ and $1/\rho_1$ for the formula for the reflection coefficient of the infinitely long partition to apply. To determine the conditions for resonance in the cavity, consider the electric field in the region between the partitions to be the sum of TE_{01} modes propagating in the positive and negative z-directions. Thus,

$$E_\theta(r, z) = AJ_1(\Gamma_1 r)e^{-j\beta_1 z} + BJ_1(\Gamma_1 r)e^{j\beta_1 z}.$$

At $z=0$, the ratio of the reflected wave $BJ_1(\Gamma_1 r)$ to the incident wave $AJ_1(\Gamma_1 r)$ must equal the reflection coefficient R. Similarly, at $z=-L$, the ratio of the reflected wave $AJ_1(\Gamma_1 r)e^{j\beta_1 L}$ to the incident wave $BJ_1(\Gamma_1 r)e^{-j\beta_1 L}$ must also equal R. Since R is equal to $e^{j\phi}$, these conditions are simply

$$\frac{B}{A} = e^{j\phi}$$

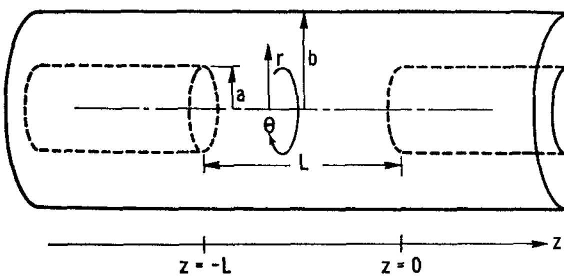

Fig. 5. Model of open-ended cavity.

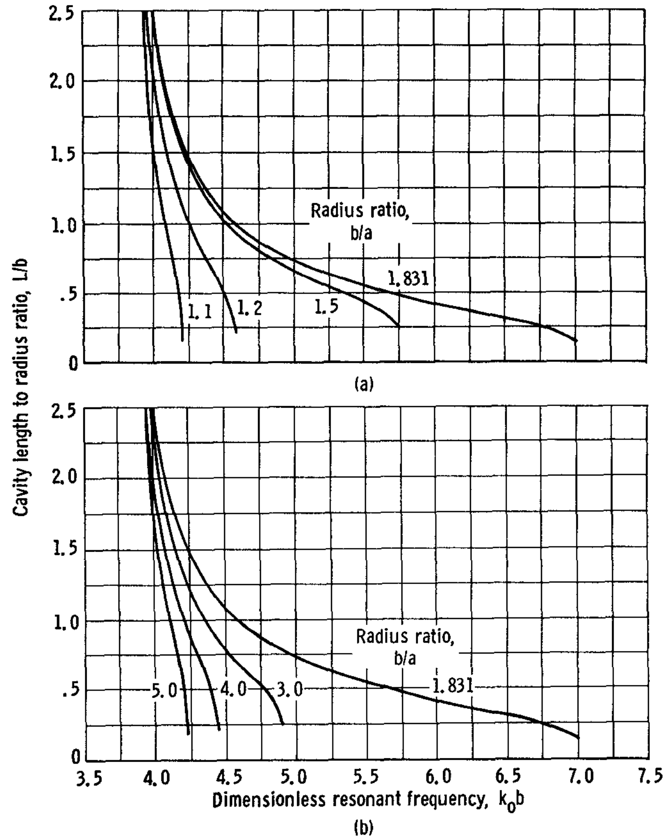

Fig. 6. Ratio of cavity length to radius against dimensionless resonant frequency for constant values of radius ratio.

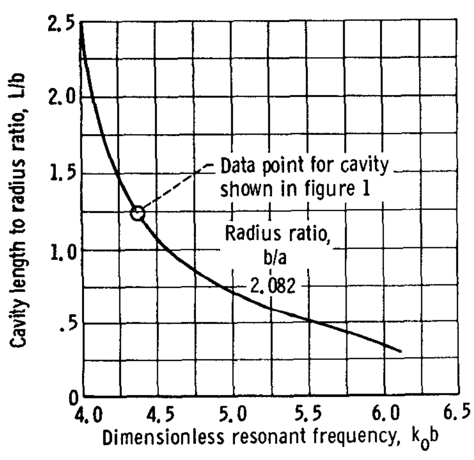

Fig. 7. Ratio of cavity length to radius against dimensionless resonant frequency for radius ratio of 2.082.

and

$$\frac{Ae^{j\beta_1 L}}{Be^{-j\beta_1 L}} = e^{j\phi}.$$

It can be easily shown that, if there is to be a nontrivial solution for amplitudes A and B, ϕ and L must be related by the equation

$$e^{j\beta_1 L} - e^{2j\phi - j\beta_1 L} = 0$$

which has the solution

$$\beta_1 L = \phi. \tag{26}$$

The required spacing between the partitions to resonate the TE_{011} mode can be found by solving (26) for L, using the values of ϕ given in Figs. 4(a) and 4(b). The spacing for the TE_{01n} resonant mode can be found by simply adding $(n-1)2\pi$ to ϕ before solving for L.

The numerical results for the TE_{011} mode are shown in Figs. 6(a), 6(b), and (7) in the form of curves of L/b against $k_0 b$ for typical values of the parameter b/a. The range of resonant frequencies over which the cavity can be tuned by varying L/b is a strong function of b/a. The value $b/a = 1.831$ gives the maximum tuning range.

The numerical results shown in Fig. 7 are for $b/a = 2.082$. This value is commonly used in practice since it corresponds to locating the cylindrical partition at the radius where the electric field of the TE_{01} mode has its maximum intensity, i.e., $J_1(\Gamma_1 r)$ has a maximum at $r = b/2.082$. This value of b/a was used in the design of the cavity shown in Fig. 1. The data point in Fig. 7 corresponding to the measured values of L/b and $k_0 b$ for this cavity indicates that the theoretical and experimental results are in good agreement.

References

[1] M. Sucher and J. Fox, *Handbook of Microwave Measurements*, 3rd ed., vol. 2. New York: Interscience, 1963, ch. 9.

[2] A. W. Adey, "Microwave refractometer cavity design," *Canad. J. Tech.*, vol. 34, pp. 519–521, March 1957.

[3] M. C. Thompson, Jr., F. E. Freethey, and D. M. Waters, "End plate modifications of X-band TE_{011} cavity resonators," *IRE Trans. on Microwave Theory and Techniques* (*Correspondence*), vol. MTT-7, pp. 388–389, July 1959.

[4] D. C. Thorn and A. W. Straiton, "Design of open-ended microwave resonant cavities," *ibid.*, pp. 389–390.

[5] R. O. Gilmer and D. C. Thorn, "Some design criteria for open-ended microwave cavities," University of New Mexico, Albuquerque, N. Mex., Tech. Rept. EE-65, June 1962.

[6] S. Ramo and J. R. Whinnery, *Fields and Waves in Modern Radio*, 2nd ed. New York: Wiley, 1953, pp. 364–366, 374–379.

[7] R. E. Collin, *Fieid Theory of Guided Waves*. New York: McGraw-Hill, 1960, ch. 10.

[8] R. Mittra and C. P. Bates, "An alternative approach to the solution of a class of Wiener-Hopf and related problems," University of Illinois, Urbana, Ill., Antenna Lab. Rept. 65-21, February 1966.

[9] B. Noble, *Methods Based on the Wiener-Hopf Technique for the Solution of Partial Differential Equations*. New York: Pergamon, 1958.

[10] A. E. Heins and S. Silver, "The edge conditions and field representation theorems in the theory of electromagnetic diffraction," *Proc. Cam. Phil. Soc.*, vol. 51, pp. 149–161, January 1955.

[11] For an example of this technique see P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, vol. 1. New York: McGraw-Hill, 1953, pp. 382–385.

Theory of Direct-Coupled-Cavity Filters

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***Abstract*—A new theory is presented for the design of direct-coupled-cavity filters in transmission line or waveguide. It is shown that for a specified range of parameters the insertion-loss characteristic of these filters in the case of Chebyshev equal-ripple characteristic is given very accurately by the formula**

$$\frac{P_0}{P_L} = 1 + h^2 T_n^2 \left[\frac{\omega_0}{\omega} \frac{\sin\left(\pi \frac{\omega}{\omega_0}\right)}{\sin \theta_0'} \right]$$

where h defines the ripple level, T_n is the first-kind Chebyshev polynomial of degree n, ω/ω_0 is normalized frequency, and θ_0' is an angle proportional to the bandwidth of a distributed lowpass prototype filter. The element values of the direct-coupled filter are related directly to the step impedances of the prototype whose values have been tabulated. The theory gives close agreement with computed data over a range of parameters as specified by a very simple formula. The design technique is convenient for practical applications.

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Introduction

A NEW TREATMENT of the classic problem of direct-coupled microwave filter design is presented. These filters consist of TEM transmission line or waveguide cavities coupled either by series capacitances or by shunt inductances, as shown in Figs. 1(a) and 1(b), respectively. It will be assumed that the waveguide or transmission line is of uniform impedance. The more general case is perhaps of less importance for economic reasons, but it has been discussed by Young [1], and the theory presented here may be extended as described by that author.